# Weighted Additive Spanners

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# Overview

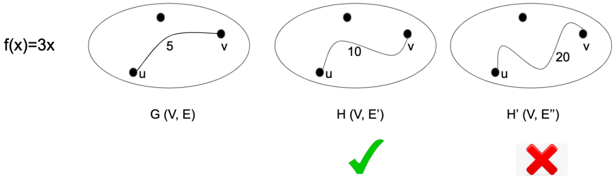
# Introduction



## Definition ($f(\cdot)$-spanner)

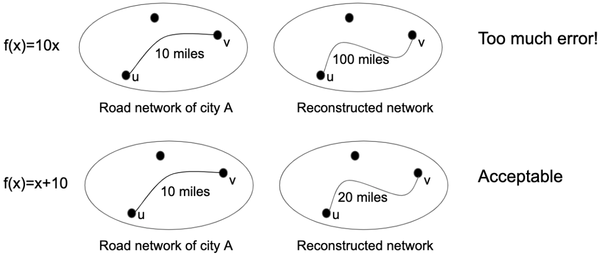
*An $f(\cdot)$-spanner of an undirected graph $G = (V, E)$ is a subgraph $H$ such that*

$$dist_H(s,t) \leq f(dist_G(s,t)), \forall s,t \in V$$



# Introduction

- If $f(d) = cd$ for some positive constant $c$, then this is called multiplicative spanner. It was introduced by Peleg and Schäffer in 1989 [10].
- If $f(d) = d + c$, then we have an additive spanner, which provide tighter distance guarantees which can be useful in some applications.




- Liestman and Shermer introduced additive spanners in 1993, and studied the concept for some classes of input graphs [9].

# Introduction



- Rules of the game: we expect that the tighter the guarantee, the more edges there might be needed.
- Multiplicative spanners follow this expectation: number of edges in a spanner grows from $n$ to $n^2$ as the stretch factor decreases.
- For additive spanners things also started that way:

| Stretch | Size | Authors |
|---|---|---|
| $+2$ | $O(n^{3/2})$ | Aingworth et al. [2] |
| $+4$ | $O(n^{7/5})$ | Chechik [5], Bodwin [4] |
| $+6$ | $O(n^{4/3})$ | Baswana et al. [3], Knudsen [8], Woodruff [11] |

- In 2017, Abboud and Bodwin [1] give examples of graphs that have no $c$-additive spanner on $O(n^{4/3-\epsilon})$ edges.

# Introduction





- Multiplicative spanners work instantly for weighted graphs.
- Previously, additive spanners were only applicable to real-world metrics that just happened to have unit edge weights.

# Problem Definition



We introduce the following problems:

**Problem (Weighted Additive Spanner)**

*Given a weighted, undirected graph $G(V, E, w)$, compute a subgraph $H$ such that*

$$dist_H(s, t) \leq dist_G(s, t) + cW, \forall s, t \in V$$

*where $W$ is the maximum weight of the edges. The subgraph $H$ is called a $cW$-spanner of $G$. The objective is to minimize the total number of edges in $H$.*

# Problem Definition



**Problem (Pairwise Weighted Additive Spanner)**

*Given a weighted, undirected graph $G(V, E, w)$, and a set of pairs of vertices $P$ compute a subgraph $H$ such that*

$$dist_H(s,t) \leq dist_G(s,t) + cW, \forall s,t \in P$$

**Problem (Subsetwise Weighted Additive Spanner)**

*Given a weighted, undirected graph $G(V, E, w)$, and a set of vertices $S \subseteq V$ compute a subgraph $H$ such that*

$$dist_H(s,t) \leq dist_G(s,t) + cW, \forall s,t \in S \times S$$

# Our Contributions



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- Almost all existing additive spanner algorithm are developed for unweighted graphs [2, 5, 3]
- To the best of our knowledge, only the model proposed by Elkin et al. [6] can be used to construct $+2$W -spanner.
- We extend the existing algorithms for unweighted graphs for weighted graphs
- However, the extensions are not straightforward

# Our Contributions (cont.)



### Theorem 1

*For any $G = (V, E, w)$ and demand pairs $P$, there is a $+2W$ pairwise spanner with $O(n|P|^{1/3})$ edges. In the all-pairs setting $P = V \times V$, the bound improves to $O(n^{3/2})$.*

### Theorem 2

*For any $G = (V, E, w)$ and demand pairs $P$, there is a $+4W$ pairwise spanner with $O(n|P|^{2/7})$ edges. In the all-pairs setting $P = V \times V$, the bound improves to $O(n^{7/5})$.*

| Unweighted | | Weighted | |
|---|---|---|---|
| Stretch | Size | Stretch | Size |
| $+2$ | $O(n^{3/2})$ | $+2W$ | $O(n^{3/2})$ |
| $+4$ | $O(n^{7/5})$ | $+4W$ | $O(n^{7/5})$ |

# Our Contributions (cont.)



## Theorem 3

*For any $G = (V, E, w)$ and demand pairs $P$, there is a $+8W$ pairwise spanner with $O(n|P|^{1/4})$ edges. In the all-pairs setting $P = V \times V$, the bound improves to $O(n^{4/3})$.*

## Theorem 4

*For any $G = (V, E, w)$ and demand pairs $P = S \times S$, there is a $+4W$ pairwise spanner with $O(n|S|^{1/2})$ edges.*

| Unweighted | | Weighted | |
|---|---|---|---|
| Stretch | Size | Stretch | Size |
| $+6$ | $O(n^{4/3})$ | $+8W$ | $O(n^{4/3})$ |
| $+2$ | $O(n\|S\|^{1/2})$ | $+4W$ | $O(n\|S\|^{1/2})$ |

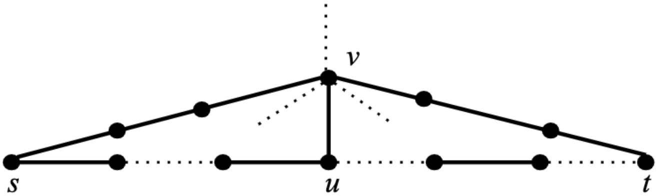
We will briefly outline the proof of Theorem 4.

# Overview of Additive Spanner Construction



- All (*unweighted*) spanner constructions start with a clustering or initialization step
- *d-initialization*: choose $d$ arbitrary edges incident to each node


- In the following step, randomly sample some nodes around a shortest path of every pair




- Finally, compute shortest path tree around each sample node, and add them

# Additive Spanners



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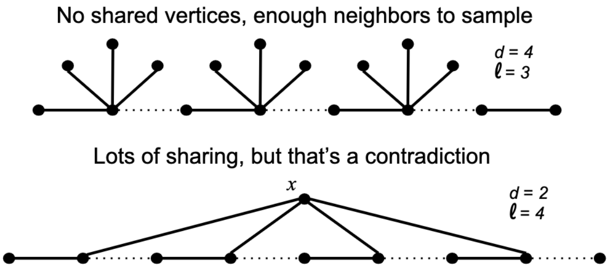
## Lemma 5

*Let $G$ be an undirected unweighted graph, let $\pi$ be a shortest path, and let $H$ be a $d$-initialization of $G$. If $\pi$ is missing $\ell$ edges in $H$, then there are $\Omega(d\ell)$ different nodes adjacent to $\pi$ in $H$.*

- Let $(u, v)$ be a missing edge of $\pi$
- Both $u$ and $v$ have degree at least $d$
- If every node is adjacent to at most one node in $\pi$, then the claim is true
- Otherwise we will over–count some nodes
- However, one node can be adjacent to at most three nodes in $\pi$

# Additive Spanners (cont.)





# Weighted Additive Spanners



- What's Harder With Weights?

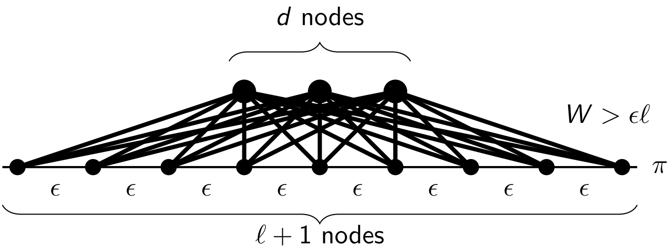



Figure: A counterexample to Lemma 5 for weighted graphs.

# Weighted Additive Spanners (cont.)



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## Lemma (5)

*Let $G$ be an undirected unweighted graph, let $\pi$ be a shortest path, and let $H$ be a $d$-initialization of $G$. If $\pi$ is missing $\ell$ edges in $H$, then there are $\Omega(d\ell)$ different nodes adjacent to $\pi$ in $H$.*

## Theorem 6

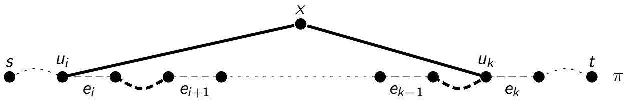
*If $H$ is a $d$-light initialization of an undirected weighted graph $G$, and there is a shortest path $\pi$ in $G$ that is missing $\ell$ edges in $H$, then there are $\Omega(d\ell)$ nodes adjacent to $\pi$ in $H$.*

# Weighted Additive Spanners (cont.)

## Lemma 7

*Let $\pi$ be a shortest path, let $x \in V$ be a node such that $x \in N^*(u_i) \cap N^*(u_k)$ for some $1 \leq i < k \leq \ell$, and consider the edges $e_i, \ldots, e_k \in M$ (the set of missing edges) with weights $w_i, \ldots, w_k$.*

*Then $w_k \geq \sum_{i'=i+1}^{k-1} w_{i'}$.*



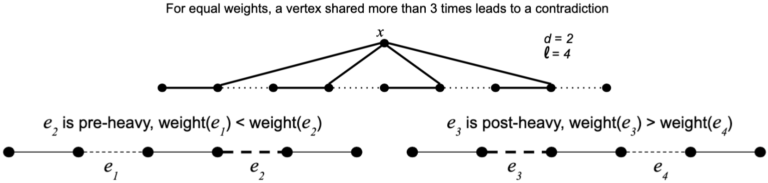

$$\sum_{i'=i}^{k-1} w_{i'} \leq \text{length}\left(\pi[u_i \rightsquigarrow u_k]\right)$$

$$\leq w(u_i, x) + w(x, u_k) \qquad (\pi[u_i \rightsquigarrow u_k] \text{ is a shortest path})$$

$$\leq w_i + w_k$$

# Weighted Additive Spanners (cont.)






**Lemma 8**

*Either more than $\frac{\ell}{2}$ edges in $M$ (the set of missing edges) are not pre-heavy, or more than $\frac{\ell}{2}$ edges in $M$ are not post-heavy.*

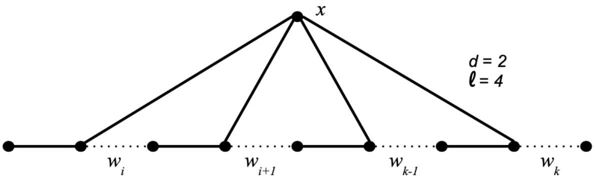
We can prove Lemma 8 using the pigeonhole principle

**Lemma 9**

*Let $\pi$ be a shortest path. For any node $x \in V$, there exist at most three nodes $u$ along $\pi$ such that $x \in N^*(u)$ and edge $(u, v) \in M$ is not pre-heavy.*

# Proof of Lemma 9






- $w_k \geq \sum_{i'=i+1}^{k-1} w_i' = w_{i+1} + \ldots + w_{k-1} \geq w_{i+1} + w_{k-1}$
- By assumption, $e_k = (u_k, v_k)$ is pre-light, so $w_{k-1} \geq w_k$
- Hence, $w_k \geq w_{i+1} + w_{k-1} \geq w_{i+1} + w_k$, or $w_{i+1} = 0$.
- Since edge weights are strictly positive, we have contradiction

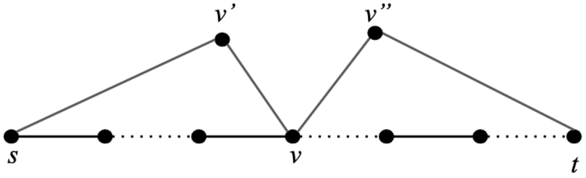
# Subsetwise $+4W$ spanner



**Theorem 10**

*For any $G = (V, E, w)$ and demand pairs $P = S \times S$, there is a $+4W$ pairwise spanner with $O(n|S|^{1/2})$ edges.*

- A pair of nodes $(s, v)$ is *near connected* if there exists $v'$ adjacent to $v$ in $H$ such that $dist_H(s, v') = dist_G(s, v')$



# Subsetwise $+4W$ spanner (cont.)



- Otherwise, add all the missing edges
- Which makes all the neighbor nodes near connected
- There are $\Omega(\ell d)$ neighbors
- And there are at most $|S|n$ pairs
- Hence, we add at most $O(|S|n/d)$ edges

# Conclusion and Future Work



- Described the main difference between weighted and unweighted additive spanners.
- Outlined the proof of $+4W$ subsetwise spanner.

| Unweighted | | Weighted | |
|---|---|---|---|
| Stretch | Size | Stretch | Size |
| $+2$ | $O(n^{3/2})$ [2] | $+2W$ | $O(n^{3/2})$ **[this paper]**, [6] |
| $+4$ | $O(n^{7/5})$ [5, 4] | $+4W$ | $O(n^{7/5})$ **[this paper]** |
| $+6$ | $O(n^{4/3})$ [3, 8, 11] | $+6W$ | ? |
| $+c$ | $\Omega(n^{4/3-\epsilon})$ [1, 7] | $+8W$ | $O(n^{4/3})$ **[this paper]** |

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