Weighted Additive Spanners

Reyan Ahmed, Greg Bodwin, Faryad Darabi Sahneh, Stephen Kobourov, and Richard Spence

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Our Contributions

Weighted Additive Spanners

Subsetwise +4W spanner

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**Introduction**

**Definition (f(·)-spanner)**

An $f(\cdot)$-spanner of an undirected graph $G = (V, E)$ is a subgraph $H$ such that

$$\text{dist}_H(s, t) \leq f(\text{dist}_G(s, t)), \forall s, t \in V$$

Where $f(x) = 3x$.

![Graphs and spans](image)
If $f(d) = cd$ for some positive constant $c$, then this is called multiplicative spanner. It was introduced by Peleg and Schäffer in 1989 [10].

If $f(d) = d + c$, then we have an additive spanner, which provide tighter distance guarantees which can be useful in some applications.

Liestman and Shermer introduced additive spanners in 1993, and studied the concept for some classes of input graphs [9].
Introduction

- Rules of the game: we expect that the tighter the guarantee, the more edges there might be needed.
- Multiplicative spanners follow this expectation: number of edges in a spanner grows from $n$ to $n^2$ as the stretch factor decreases.
- For additive spanners things also started that way:

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<thead>
<tr>
<th>Stretch</th>
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<th>Authors</th>
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<tbody>
<tr>
<td>+2</td>
<td>$O(n^{3/2})$</td>
<td>Aingworth et al. [2]</td>
</tr>
<tr>
<td>+4</td>
<td>$O(n^{7/5})$</td>
<td>Chechik [5], Bodwin [4]</td>
</tr>
<tr>
<td>+6</td>
<td>$O(n^{4/3})$</td>
<td>Baswana et al. [3], Knudsen [8], Woodruff [11]</td>
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- In 2017, Abboud and Bodwin [1] give examples of graphs that have no $c$-additive spanner on $O(n^{4/3-\epsilon})$ edges.
Introduction

- Multiplicative spanners work instantly for weighted graphs.
- Previously, additive spanners were only applicable to real-world metrics that just happened to have unit edge weights.
We introduce the following problems:

**Problem (Weighted Additive Spanner)**

*Given a weighted, undirected graph $G(V, E, w)$, compute a subgraph $H$ such that*

$$\text{dist}_H(s, t) \leq \text{dist}_G(s, t) + cW, \forall s, t \in V$$

*where $W$ is the maximum weight of the edges. The subgraph $H$ is called a $cW$-spanner of $G$. The objective is to minimize the total number of edges in $H$.***
Problem Definition

Problem (Pairwise Weighted Additive Spanner)

Given a weighted, undirected graph $G(V, E, w)$, and a set of pairs of vertices $P$ compute a subgraph $H$ such that

$$\text{dist}_H(s, t) \leq \text{dist}_G(s, t) + cW, \forall s, t \in P$$

Problem (Subsetwise Weighted Additive Spanner)

Given a weighted, undirected graph $G(V, E, w)$, and a set of vertices $S \subseteq V$ compute a subgraph $H$ such that

$$\text{dist}_H(s, t) \leq \text{dist}_G(s, t) + cW, \forall s, t \in S \times S$$
Our Contributions

- Almost all existing additive spanner algorithm are developed for unweighted graphs [2, 5, 3]
- To the best of our knowledge, only the model proposed by Elkin et al. [6] can be used to construct $+2W$ -spanner.
- We extend the existing algorithms for unweighted graphs for weighted graphs
- However, the extensions are not straightforward
Our Contributions (cont.)

Theorem 1

For any $G = (V, E, w)$ and demand pairs $P$, there is a $+2W$ pairwise spanner with $O(n|P|^{1/3})$ edges. In the all-pairs setting $P = V \times V$, the bound improves to $O(n^{3/2})$.

Theorem 2

For any $G = (V, E, w)$ and demand pairs $P$, there is a $+4W$ pairwise spanner with $O(n|P|^{2/7})$ edges. In the all-pairs setting $P = V \times V$, the bound improves to $O(n^{7/5})$.

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<td>$O(n^{7/5})$</td>
</tr>
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</table>
Our Contributions (cont.)

**Theorem 3**

For any $G = (V, E, w)$ and demand pairs $P$, there is a $+8W$ pairwise spanner with $O(n|P|^{1/4})$ edges. In the all-pairs setting $P = V \times V$, the bound improves to $O(n^{4/3})$.

**Theorem 4**

For any $G = (V, E, w)$ and demand pairs $P = S \times S$, there is a $+4W$ pairwise spanner with $O(n|S|^{1/2})$ edges.

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<td>Stretch</td>
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<td>$O(n</td>
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We will briefly outline the proof of Theorem 4.
Overview of Additive Spanner Construction

- All (*unweighted*) spanner constructions start with a clustering or initialization step.
- *d-initialization:* choose *d* arbitrary edges incident to each node.
- In the following step, randomly sample some nodes around a shortest path of every pair.
- Finally, compute shortest path tree around each sample node, and add them.
Lemma 5

Let $G$ be an undirected unweighted graph, let $\pi$ be a shortest path, and let $H$ be a $d$-initialization of $G$. If $\pi$ is missing $\ell$ edges in $H$, then there are $\Omega(d\ell)$ different nodes adjacent to $\pi$ in $H$.

- Let $(u, v)$ be a missing edge of $\pi$
- Both $u$ and $v$ have degree at least $d$
- If every node is adjacent to at most one node in $\pi$, then the claim is true
- Otherwise we will over-count some nodes
- However, one node can be adjacent to at most three nodes in $\pi$
Additive Spanners (cont.)

No shared vertices, enough neighbors to sample

Lots of sharing, but that’s a contradiction
Weighted Additive Spanners

- What’s Harder With Weights?

![Diagram](image)

Figure: A counterexample to Lemma 5 for weighted graphs.
Lemma (5)

Let $G$ be an undirected unweighted graph, let $\pi$ be a shortest path, and let $H$ be a $d$-initialization of $G$. If $\pi$ is missing $\ell$ edges in $H$, then there are $\Omega(d\ell)$ different nodes adjacent to $\pi$ in $H$.

Theorem 6

If $H$ is a $d$-light initialization of an undirected weighted graph $G$, and there is a shortest path $\pi$ in $G$ that is missing $\ell$ edges in $H$, then there are $\Omega(d\ell)$ nodes adjacent to $\pi$ in $H$. 
Lemma 7

Let $\pi$ be a shortest path, let $x \in V$ be a node such that $x \in N^*(u_i) \cap N^*(u_k)$ for some $1 \leq i < k \leq \ell$, and consider the edges $e_i, \ldots, e_k \in M$ (the set of missing edges) with weights $w_i, \ldots, w_k$.

Then $w_k \geq \sum_{i'=i+1}^{k-1} w_{i'}$.

$$
\sum_{i'=i}^{k-1} w_{i'} \leq \text{length}\left(\pi[u_i \leadsto u_k]\right)
\leq w(u_i, x) + w(x, u_k) \quad (\pi[u_i \leadsto u_k] \text{ is a shortest path})
\leq w_i + w_k
$$
Weighted Additive Spanners (cont.)

Lemma 8

Either more than \( \frac{\ell}{2} \) edges in \( M \) (the set of missing edges) are not pre-heavy, or more than \( \frac{\ell}{2} \) edges in \( M \) are not post-heavy.

We can prove Lemma 8 using the pigeonhole principle.

Lemma 9

Let \( \pi \) be a shortest path. For any node \( x \in V \), there exist at most three nodes \( u \) along \( \pi \) such that \( x \in N^*(u) \) and edge \( (u, v) \in M \) is not pre-heavy.
Proof of Lemma 9

- \( w_k \geq \sum_{i'=i+1}^{k-1} w_i' = w_{i+1} + \ldots + w_{k-1} \geq w_{i+1} + w_{k-1} \)
- By assumption, \( e_k = (u_k, v_k) \) is pre-light, so \( w_{k-1} \geq w_k \)
- Hence, \( w_k \geq w_{i+1} + w_{k-1} \geq w_{i+1} + w_k \), or \( w_{i+1} = 0 \).
- Since edge weights are strictly positive, we have contradiction.
Theorem 10

For any $G = (V, E, w)$ and demand pairs $P = S \times S$, there is a $+4W$ pairwise spanner with $O(n|S|^{1/2})$ edges.

- A pair of nodes $(s, v)$ is near connected if there exists $v'$ adjacent to $v$ in $H$ such that $dist_H(s, v') = dist_G(s, v')$.
Otherwise, add all the missing edges
- Which makes all the neighbor nodes near connected
- There are $\Omega(\ell d)$ neighbors
- And there are at most $|S|n$ pairs
- Hence, we add at most $O(|S|n/d)$ edges
Conclusion and Future Work

- Described the main difference between weighted and unweighted additive spanners.
- Outlined the proof of $+4W$ subsetwise spanner.

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<td>$+2W$</td>
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<td>$O(n^{7/5})$ [5, 4]</td>
<td>$+4W$</td>
</tr>
<tr>
<td>$+6$</td>
<td>$O(n^{4/3})$ [3, 8, 11]</td>
<td>$+6W$</td>
</tr>
<tr>
<td>$+c$</td>
<td>$\Omega(n^{4/3} - \epsilon)$ [1, 7]</td>
<td>$+8W$</td>
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Future Work
Amir Abboud and Greg Bodwin.
The 4/3 additive spanner exponent is tight.  

Donald Aingworth, Chandra Chekuri, Piotr Indyk, and Rajeev Motwani. 
Fast estimation of diameter and shortest paths (without matrix multiplication). 

Surender Baswana, Telikepalli Kavitha, Kurt Mehlhorn, and Seth Pettie.  
Additive spanners and $(\alpha, \beta)$-spanners.  

Greg Bodwin. 
A note on distance-preserving graph sparsification.  

Shiri Chechik.
New additive spanners.


Michael Elkin, Yuval Gitlitz, and Ofer Neiman.
Almost shortest paths and PRAM distance oracles in weighted graphs.

Shang-En Huang and Seth Pettie.
Lower bounds on sparse spanners, emulators, and diameter-reducing shortcuts.

Mathias Bæk Tejs Knudsen.
Additive spanners: A simple construction.
